短文與短評
NOTES AND CORRESPONDENCE

Comparison of McGinley's and O'Brien's Variational Optimization Formulation for the Computation of Vertical Velocity

WEN-JEY LIANG

Institute of Physics, Academia Sinica, and Dept. of Mech. Eng., National Taiwan University

(Manuscript received 28 February 1977, in revised form 5 April 1977)

ABSTRACT

In this study, the comparison of McGinley's and O'Briens variational optimization scheme is discussed. A generalized variational formulation is defined to include both schemes, and the Fourier analysis is utilized to analyze its results. The conclusions show that if the error variance of the velocity field is proportional to that of the divergence field, two schemes are exactly the same, or, the differences of their responses depend on the functions of error variance.

1. Introduction

A current important problem in meteorology is the estimation of the distribution of vertical velocity in the atmosphere. Because the routine observation of vertical velocity is not available, it is commonly computed from the horizontal wind velocity, the pressure, or the temperature distribution utilizing the continuity equation (the kinematic method), the emega equation, or the adiabatic equation, respectively. Although each method has its meaning, kinematic method seems worthist to mention because the hydrostatic assumption is a good approximation for the atmospheric motion in many situations. However, errors in the wind observations often tend to accumulate during the vertical integration that there is little confidence in the use of the computed vertical velocity without considerable corrections.

The errors which appear in the computed vertical velocity field can be classified as "systematic errors" and "random errors". The systematic errors occur primarily because of inconsistencies between the observed field and the dynamical model considered. The random errors may be introduced because of inaccuracies of measurements, spatial irregularity of observation points, and by interpolation of values from stations to grid points. In the past years, many methods have been proposed to correct those errors. One of those sophisticated methods is Sasaki's variational optimization approach. This method provides an important advantage by incorporating dynamic, kinematic, statistical and other conditions in data management. Consequently, systematic errors as well as random errors can be suppressed such that the optimized data are consistent with the dynamical model considered. For these purposes, McGinley (1973) and O'Brien (1970) developed two individual schemes according to the Sasaki's approach. Each scheme has its mathematical and physical basis and is succeeded in its applications. However, it is believed that the theoretical comparison of two schemes is essentially important for selecting a better one. In these study, a generalized variational formulation is defined to include both schemes, and the Fourier analysis is performed to analyze its results and those of the McGinley's and the O'Brien's scheme.

2. Generalized Variational Formulation

The generalized functional is defined as

$$I=\iiint_{pyx}\{\alpha\,|\,\mathbf{V}-\tilde{\mathbf{V}}\,|^2+\beta(D-\tilde{D})^2\}\,dx\,dy\,dp, \quad (1)$$

and the strong constrant is

$$\frac{\partial\omega}{\partial p}+\nabla\cdot\mathbf{V}=0, \quad (2)$$

where $\mathbf{V}$ and $\tilde{\mathbf{V}}$ are observed and optimized wind, respectively, α and β are specified parameters and are functions of p, and

$$D\equiv\nabla\cdot\mathbf{V},$$

$$\tilde{D}\equiv\nabla\cdot\tilde{\mathbf{V}}=-\frac{\partial\tilde{\omega}}{\partial p},$$

and

$$\nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right).$$

It is obvious that (1) reduces to the McGinley's or the O'Brien's formulation as β or α is equal to zero, respectively. Indeed, the original McGinley's scheme is the case as α is equal to 1. However, we will refer the case of β equal to zero as McGinley's scheme even if α is arbitrary.

After setting the variation of the functional I to be zero, we obtain the associated Euler-Lagrange equations:

$$\alpha(\mathbf{V}-\tilde{\mathbf{V}})=\beta\nabla(D-\tilde{D})+\nabla\lambda, \quad (3)$$

$$\frac{\partial\lambda}{\partial p}=0, \quad (4)$$

where λ is the Lagrange multiplier. Eq. (4) implies that λ is a function of x and y alone. Let us define the correction divergence, D^*, as

$$D^*=D-\tilde{D}, \quad (5)$$

and take divergence operator on (3), then, we obtain:

$$\nabla^2\lambda=(\alpha-\beta\nabla^2)D^* \quad (6)$$

The integration of (2) and the substitution of (5) into the result equation leads to

$$\omega_t-\tilde{\omega}_t=\int_{p_t}^{p_s}D^*dp, \quad (7)$$

where

$$\omega_t=\omega_s+\int_{p_t}^{p_s}Ddp$$

$$\tilde{\omega}_t=\tilde{\omega}_s+\int_{p_t}^{p_s}\tilde{D}dp,$$

p_s and p_t are pressures at the surface and at the top of atmosphere, respectively, and ω_s is assumed to be equal to $\tilde{\omega}_s$.

Now, we will assume D^* to be a separable form in x, y and p, i. e.,

$$D^*=A(x,y)B(p). \quad (8)$$

Then, the substitution of (8) into (7) leads to

$$A=(\omega_t-\tilde{\omega}_t)/\bar{B}, \quad (9)$$

where

$$(\bar{\ })\equiv\int_{p_t}^{p_s}(\ )dp.$$

Since λ is independent of p, the Fourier transformation in x and y of (6) implies

$$\frac{-B(\alpha+\beta k^2)}{k^2}=C,$$

or

$$B=-Ck^2\sigma, \quad (10)$$

where C is a constant, k^2 is the sum of the square of the wave numbers in x and y directions, σ is the error variance and is defined as

$$\sigma=\frac{1}{\alpha+\beta k^2}$$

The integration of (10) leads to

$$C=-\frac{\bar{B}}{k^2\bar{\sigma}}, \quad (11)$$

and the substitution of (11) into (10) leads to

$$B=\frac{\sigma\bar{B}}{\bar{\sigma}} \quad (12)$$

Therefore, from (8), (9) and (12), we have

$$D^*=\frac{\sigma}{\bar{\sigma}}(\omega_t-\tilde{\omega}_t), \quad (13)$$

and the optimized vertical velocity and divergence are

$$D=\tilde{D}+\frac{\sigma}{\bar{\sigma}}(\omega_t-\tilde{\omega}_t), \quad (14)$$

$$\omega=\omega_t-\int_{p_t}^{p}\tilde{D}dp-\frac{(\omega_t-\tilde{\omega}_t)}{\bar{\sigma}}\int_{p_t}^{p}\sigma dp. \quad (15)$$

3. Special Cases

a. If α is equal to zero, $\sigma/\bar{\sigma}$ is reduced to

$$\frac{\sigma}{\bar{\sigma}}=\frac{1}{\beta}\Big/\int_{p_t}^{p}\frac{dp}{\beta}. \quad (16)$$

Furthermore, if $1/\beta$ is assumed to be a linear function of p, (16) and (14) become

$$\frac{\sigma}{\bar{\sigma}}=\frac{2(p_s-p)}{p_s^2-p_t^2}, \quad (17)$$

and

$$D=\tilde{D}+\frac{2(p_s-p)}{p_s^2-p_t^2}(\omega_t-\tilde{\omega}_t). \quad (18)$$

It is obvious that (17) and (18) is equivalent to the O'Brien's scheme.

b. If β is equal to zero, α is equal to 1, and ω_t is assumed to be zero, $\sigma/\tilde{\sigma}$ is reduced to

$$\frac{\sigma}{\tilde{\sigma}}=\frac{1}{p_s-p_t}.$$

and Eq. (14) and (6) become

$$D-\tilde{D}=\frac{\tilde{\omega}_t}{p_s-p_t}, \quad (19)$$

$$\nabla^2\lambda=-\frac{\tilde{\omega}_t}{p_s-p_t}.$$

Eq. (19) shows that this case is equivalent to the McGinley's scheme.

c. If α is proportional to β, i. e.,

$$\alpha=\gamma\beta,$$

then,

$$\frac{\sigma}{\tilde{\sigma}}=\frac{1}{\beta\int_{p_t}^{p_s}\frac{dp}{\beta}}=\frac{1}{\alpha\int_{p_t}^{p_s}\frac{dp}{\alpha}} \quad (20)$$

Eq. (20) shows that the McGinley's and the O'Brien's formulation are essentially the same if the error variance of the velocity field is proportional to that of the divergence field.

4. Conclusion

The above discussions show that the McGinley's and the O'Brien's variational optimization scheme are the special cases of the generalized variational formulation defined in (1). It is also shown that if the error variance of the velocity field is proportional to that of the divergence field, i. e. $\alpha\propto\beta$, McGinley's scheme is exactly the same as O'Brien's scheme.

If α and β are independent, the generalized variational formulation shows a significant characteristics. The correction divergence, D^*, depends on wave number, k, as well as the error variances corresponding to α and β. If we nondimensionalize the formulus utilizing, say, 1000 km, 10 m/sec, $10^{-6}\,\text{sec}^{-1}$, p_s and 1 μb/sec as the length, velocity, divergence, pressure and vertical velocity scales, respectively, (14) becomes

$$D'=\tilde{D}'+\frac{\sigma'}{\tilde{\sigma}'}(\omega_t'-\tilde{\omega}_t'), \quad (21)$$

where

$$\frac{\sigma'}{\tilde{\sigma}'}=\frac{1}{\alpha'+\beta'p'^2}\bigg/\int_{p_t'}^{1}\frac{dp'}{\alpha'+\beta'k'^2}, \quad (22)$$

and $(\ )'$ denotes the nondimensional variables. Because the errors in divergence field are several times larger than those in velocity field, α' should be the quare of this times larger than β'. If we assume that α' and β' are proportional to $(p')^{-m}$ and $(p')^{-n}$, respectively, i. e.,

$$\alpha'=a(1-p')^{-m},$$

and

$$\beta=b(1-p')^{-n},$$

Eq. (22) becomes

$$\frac{\sigma'}{\tilde{\sigma}'}=\frac{1}{a(1-p')^{-m}+b(1-p')^{-n}k'^2}\bigg/\int_{p_t'}^{1}\frac{dp'}{a(1-p')^{-m}+b(1-p')^{-n}k'^2}. \quad (23)$$

As the wave number varies from zero to infinite, $\sigma'/\tilde{\sigma}'$ varies from $(m+1)(1-p')^m$ to $(n+1)(1-p')^n$ as p_t' is zero. Therefore, if n is larger than m, the correction of divergence of short waves is larger than that of long waves at higher levels and is smaller at lower levels. And, if n is equal to m, the correction divergence is independent of wave number. Furthermore, since $\sigma'/\tilde{\sigma}'$ is of the order of 1, the nondimensional correction divergence is zero at surface, is of the order 1 at the top of atmosphere, and is between these two values at any particular altitude depending on the functions of error variance.

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馬氏與歐氏垂直速度變分佳化求法之比較

梁 文 傑

中央研究院物理研究所，臺灣大學機械系

摘 要

本文將比較馬氏 (McGinley, 1973) 與歐氏 (O'Brien, 1970) 變分佳化法求垂直速度之異同，文中先將二氏之變分公式通化 (Generalization) 使二氏之公式成爲通化後公式之特例，並以福利葉 (Fourier) 轉換法分析在不同情況下，馬氏與歐氏變分佳化法對各種波長之影響。分析指出若速度場 (Velocity field) 與輻散場 (Divergence field) 之誤差變量 (Error variance) 成比例， 則二氏之方法完全相同，否則其相異則因所選取之誤差變量函數而定。